Math 19a, readings 15.2 and 16.1 review

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1 Past Temperatures Directly from the Greenland Ice Sheet

This article is "predicting the past" from the present. The authors drilled holes in Greenland's thickest ice (3000 m deep!) and measured the temperatures down through the hole. The idea is that temperature propagates through the ice the same way particles diffuse through a substance (heat equation and diffusion equation are the same), so changes on the surface of the ice sheet propagate downwards with distance and time. Knowing the present temperatures along the hole is like knowing the boundary conditions (e.g. $T(0, z)$), which enables us to solve uniquely any diffusion-advection equation, and so find $T(x, t)$ for all x, t and in particular $T(0, t)$ for all t which would be the temperature at the surface for any time in the past. (Here T is the temperature, z is the depth, t time) If we assume this simple picture, we can write a diffusion (heat) equation for $T(t, z)$:

$$\rho c \frac{\partial}{\partial t} T = \frac{\partial}{\partial z}\left(K \frac{\partial}{\partial z} T\right) + f, \tag{1}$$

where ρ is the density of ice (and a function of depth), c is the ice heat capacity (and depends on the temperature T), K is the thermal conductivity (and depends on T and ρ) and $f(z)$ is the heat production term (from earth - geothermal heat).

This model, however, does not account for the fact that ice is not stationary - it flows, however slow it be. We need to add another variable - x, which is the horizontal distance, and an advection term. The ice is moving with horizontal velocity v_x and vertical velocityv_z and this movement contributes an advection term to our equation. Having x, we should take into account that temperature diffuses horizontally also, so the diffusion term will have an additional $\frac{\partial}{\partial x}\left(K \frac{\partial}{\partial x} T\right)$ term in it, giving us the full model:

$$\rho c \frac{\partial}{\partial t} T = \frac{\partial}{\partial x}\left(K \frac{\partial}{\partial x} T\right) + \frac{\partial}{\partial z}\left(K \frac{\partial}{\partial z} T\right) - \rho c \left(v_x \frac{\partial}{\partial x} T + v_z \frac{\partial}{\partial z} T\right) + f. \tag{2}$$

Even though it is impossible to find an explicit solution to this equation, it is important to note that this equation is again fully predictive if we know the boundary condition $T(0, z)$ (i.e. what they measured down through the hole), so in theory we know $T(t, z)$ and in particular $T(t, 0)$ - the air temperatures through the past times. The authors find a solution for $T(t, z)$ via statistical methods (they use an algorithm which generates random functions for $T(t, z)$, determines how far they are from being a solution, and averages out according to the output).

2 Fishing for answers: Deep Trawls Leave Destruction in Their Wake - But for How Long?

Deep trawls drag nets along the bottom of the sea in their quest for lobsters (or other bottom-dwelling delicacies). These nets however don't discriminate and destroy not only the lobster population but even species like tube worms, as they turn over the stones worms attach to. In general the trawls plough the sea bottom, homogenizing it and destroying habitat for species in the low ends of the food chain. That's why the fish and lobster populations decreases not only because of fishermen's catch but also because their food (e.g. the same tube worms) get destroyed. Janet Raloff (the author) discusses these issues and tries to raise an alarm of how big deep trawls' impact is.

We, however, are interested in modeling lobster population in the situation when deep trawls catch everything in certain areas. Chapter 16 discusses a model for the lobster population when there is a strip $(0 < x < R)$, where deep trawls are not allowed. The model is

$$\frac{\partial}{\partial t}u = \mu \frac{\partial^2}{\partial x^2}u + ru, \tag{3}$$

where $u(t, x)$ is the lobster population at time t and position x. Here r is the growth rate of the lobster population on its own, and μ is a lobster mobility rate (diffusion coefficient). This equation comes from the fact that the rate of change of lobsters = (lobsters which diffuse in - lobsters diffusing out) + (lobsters born - lobsters who died). This of course assumes that lobsters move randomly (like particles).

For this paper we can discuss a different situation - suppose that trawls have been only allowed to plough in a given strip of length R $(0 < x < R)$ before $t = 0$, then how are the lobsters going to repopulate the devastated stripe. The equation is the same, what changes are the initial conditions. The deep trawls wiped out all lobsters in that strip at time 0, while everywhere else the population remained the same as before and is also homogenous (i.e. the same at every location, here it's 1), so

$$u(0, x) = \begin{cases} 0 & , 0 \le x \le R \\ 1 & , x < 0 \text{ or } x > R \end{cases} \tag{4}$$

According to chapter 15.2 (page 227) the general solution to the diffusion equation (3) is

$$u(t, x) \quad = \quad \frac{1}{(4\pi\mu t)^{1/2}} e^{rt} \int_{-\infty}^{\infty} u(0, s) e^{-(x-s)^2/4\mu t} ds, \tag{5}$$

where we can substitute (4) for $u(0, s)$ and then $a = s - x$ to obtain the solution given on page 254 (note the typo there, a factor of e^{rt} is missing!)

$$u(t, x) = \frac{e^{rt}}{(4\pi\mu t)^{1/2}} \int_{x}^{\infty} e^{-a^2/4\mu t} da + \frac{e^{rt}}{(4\pi\mu t)^{1/2}} \int_{R-x}^{\infty} e^{-a^2/4\mu t} da. \tag{6}$$